

## A growth-model puzzle solved

In class the other day we were puzzled by this question: Why should it be that, in the Solow growth model, the steady-state levels of capital per worker and output per worker are inversely related to the growth rate of the labor force? Actually, it turns out that this is a simple consequence of the arithmetic of the steady state, as explained below.

First, a brief preliminary. In the MRW version of the Solow model, they have technology improving steadily at some rate  $g$ , and in that context the steady-state condition is that capital stock and output per *productivity-adjusted* worker remain constant ( $K/AL$  and  $Y/AL$ ). In relation to our puzzle, though, technological change is not really germane. We're interested in comparing two hypothetical economies that are the same in all respects except for the rate of growth of the labor force. If technology is improving over time, that will benefit both economies, but it won't make a difference between them. I will therefore simplify matters by assuming there's no ongoing technological change (i.e. in mathematical terms I'll set  $A(t) = 1$  for all time). In this case the steady-state condition is that capital stock and output *per worker*,  $K/L$  and  $Y/L$ , remain constant over time.

To fix ideas, let's use some numbers. Consider two economies, A and B, observed at some point in time (a "snapshot"). Each has a workforce,  $L$ , of 1 million people, and each is producing an output,  $Y$ , of 10 million (in some suitable unit). Both economies have a rate of saving (and investment), as a fraction of output, of  $s = .10$ , and their capital stocks are subject to depreciation at a common rate  $\delta = .05$ .

Thus both economies are consuming 9 million of their output, and investing 1 million.

Here's the difference: Economy A has an unchanging workforce (growth rate of zero), while in economy B the workforce is growing at, say, 4 percent per year.

Now, let's suppose that economy A is in its steady state. The investment of 1 million is just maintaining their capital stock. Total depreciation is  $\delta K$ , and all their investment is "eaten up" by depreciation, so

$$1 \text{ million} = \delta K = sY$$

and we can calculate the steady-state capital stock as

$$K^* = \frac{sY}{\delta} = \frac{1 \text{ million}}{.05} = 20 \text{ million}$$

Each year, 1 million of the capital stock of 20 million wears out, and the annual investment of 1 million just replaces that portion. The steady-state capital stock per worker is 20.

Could the situation we have just described, with a capital/labor ratio of 20, be a steady state for economy B too? No. If investment of 1 million is just enough to cover the depreciation on a capital stock of 20 million, then it is clearly *not* enough to grow the capital stock in line with growth in the labor force of 4 percent. Starting from the snapshot position, capital per worker must fall in economy B. When will it stop falling? When the capital stock per worker has shrunk sufficiently that this condition holds: investment,  $sY$ , minus depreciation,  $\delta K$ , leaves a remainder that is enough to grow  $K$  in line with the work force (i.e. a remainder of  $.04K$ ).

To get such a remainder, we clearly require that total depreciation,  $\delta K$ , be smaller than in economy A. But given a common *rate* of depreciation,  $\delta$ , the only way to shrink the total depreciation is to have a smaller capital stock. QED.