Notes on Mankiw, Romer and Weil (continued)

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In the first notes, we got as far as deriving the following MRW equation.

$$\dot{k} = sk^{\alpha} - (n + g + \delta)k \tag{4}$$

Now let's work through the rest of their Section I.

Equation (4) relates the time-derivative of $k = K/AL$ to the level of that variable at each point in time. To find the "steady state" we set $\dot{k} = 0$:

$$0 = sk^{\alpha} - (n + g + \delta)k$$

and find the value of k (call it k^*) that solves the resulting equation.

$$sk^{*\alpha} = (n + g + \delta)k^*$$

$$\frac{k^{*\alpha}}{k^*} = \frac{n + g + \delta}{s}$$

$$k^{*\alpha - 1} = \frac{n + g + \delta}{s}$$

$$k^{*1-\alpha} = \frac{s}{n + g + \delta}$$

So

$$k^* = \left(\frac{s}{n + g + \delta}\right)^{1/(1-\alpha)} \tag{5}$$

How can we be sure that k evolves towards this steady-state value? Well, suppose $\dot{k}$ is positive. To confirm convergence towards the steady state, we need to show that $\dot{k}$ will tend to fall to zero as k itself grows. Look back at equation (4): the positive term on the right-hand side of that equation is sk^{α}, while the negative term is $(n + g + \delta)k$. Since $0 < \alpha < 1$, as k grows the negative term in k grows faster than the positive one in k^{α}, which must eventually drive the growth of k to zero.

MRW now note (p. 410) that the "central predictions of the Solow model concern the impact of saving and population growth on real income." To bring these to the fore, they substitute the expression for the steady-state k value from (5) into the production function, to get their equation (6). Let's see how this is done.

Recall that the production function was $Y = K^{\alpha}(AL)^{1-\alpha}$, which, as we saw in the first set of notes, implied $y = k^{\alpha}$ where $y = Y/AL$ and $k = K/AL$. So we can write, for the steady state,

$$y = \frac{Y}{AL} = k^{\alpha} = \left(\frac{s}{n + g + \delta}\right)^{\alpha/(1-\alpha)}$$

$$\frac{Y}{L} = A\left(\frac{s}{n + g + \delta}\right)^{\alpha/(1-\alpha)}$$

Now, taking logs ("ln" indicates a natural or base-e log), we get:

$$\ln\frac{Y}{L} = \ln A + \frac{\alpha}{1-\alpha}\ln\left(\frac{s}{n + g + \delta}\right)$$

$$\ln\frac{Y}{L} = \ln A + \frac{\alpha}{1-\alpha}\ln(s) - \frac{\alpha}{1-\alpha}\ln(n + g + \delta)$$

To get to MRW's equation (6) we just have to recognize that, since $A = A(0)e^{gt}$,

$$\ln A = \ln A(0) + gt$$

(This is an important general point: if a variable (here A) is growing exponentially, the natural log of the variable grows linearly.)