Notes on the math of IS–LM

Our IS–LM model is represented by the following equations, in general functional form:

$$
\begin{aligned}
Y &= C + I + G & & (1)\\
C &= C(Y - T) & 0 < C' < 1 \quad & (2)\\
T &= T(Y) & 0 < T' < 1 \quad & (3)\\
I &= I(r) & I' < 0 \quad & (4)\\
\tfrac{M}{P} &= L(Y, r) & L_Y > 0,\ L_r < 0 \quad & (5)\\
Y &= F(N) & F' > 0 \quad & (6)\\
\tfrac{W}{P} &= F'(N) & F' > 0,\ F'' < 0 \quad & (7)
\end{aligned}
$$

The variables are defined as follows:

Y: real GDP	C: real consumption	I: real investment
G: real government spending	T: real tax revenue	r: nominal interest rate
M: nominal money supply	P: aggregate price level	N: employment
W: nominal wage		

A 'local linearization' of the model is obtained by total differentiation, which yields the following equations:

$$
\begin{aligned}
dY &= dC + dI + dG & \text{(i)}\\
dC &= C'(dY - dT) & \text{(ii)}\\
dT &= T'dY & \text{(iii)}\\
dI &= I'dr & \text{(iv)}\\
\tfrac{1}{P}dM - \tfrac{M}{P^2}dP &= L_Y dY + L_r dr & \text{(v)}\\
dY &= F'dN & \text{(vi)}\\
\tfrac{1}{P}dW - \tfrac{W}{P^2}dP &= F''dN & \text{(vii)}
\end{aligned}
$$

Equations (i)–(vii) may be represented in vector–matrix form. Suppose we collect the endogenous variables—dY, dC, dT, dI, dr, dP and dN—on the left, and put the exogenous variables—dG, dM and dW—on the right. Then the system will look like this:

$$
\begin{bmatrix}
1 & -1 & 0 & -1 & 0 & 0 & 0\\
-C' & 1 & C' & 0 & 0 & 0 & 0\\
-T' & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -I' & 0 & 0\\
L_Y & 0 & 0 & 0 & L_r & \frac{M}{P^2} & 0\\
1 & 0 & 0 & 0 & 0 & 0 & -F'\\
0 & 0 & 0 & 0 & 0 & \frac{W}{P^2} & F''
\end{bmatrix}
\begin{bmatrix} dY\\ dC\\ dT\\ dI\\ dr\\ dP\\ dN \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & \frac{1}{P} & 0\\
0 & 0 & 0\\
0 & 0 & \frac{1}{P}
\end{bmatrix}
\begin{bmatrix} dG\\ dM\\ dW \end{bmatrix}
$$

(You may care to check and see if I have this right!) This system is of the general form

$$A\mathbf{y} = B\mathbf{x}$$

In principle we could solve for the reduced form of the model (which gives us the full effects on all of the endogenous variables of a change in the exogenous variables) thus:

$$A^{-1}A\mathbf{y} = A^{-1}B\mathbf{x}$$

$$\mathbf{y} = A^{-1}B\mathbf{x}$$

But of course inverting A would be no mean feat.

Cramer's rule provides a helping hand here, but even for Cramer we don't want to be taking the determinant of an 7×7 matrix. We need to boil the model down a bit. We will have a manageable determinant if we can get it down to 3×3. That means eliminating four of the endogenous variables. Which are good candidates for elimination? Well, both C and T are uniquely related to Y: we know that if the latter goes up (down) then so will C and T. So let's get rid of these variables—the explicit inclusion of dC and dT in the system is not telling us anything we couldn't figure out without them. Similarly, I is uniquely related (negatively) to r, and so may be eliminated. Finally, N is uniquely related to Y via the production function, so N can go. That leaves dY, dr and dP as the endogenous variables. We could eliminate dP too, but it might be handy to keep it in the system.

The eliminations go as follows. First we can use (ii), (iii) and (iv) in (i) to yield

$$[1 - C'(1 - T')]dY = I'\,dr + dG$$

and (vi) in (vii) to get

$$\frac{1}{P}\,dW - \frac{W}{P^2}\,dP = \frac{F''}{F'}\,dY$$

The above two relationships, along with (v), can be put into vector–matrix form as

$$\begin{bmatrix} 1 - C'(1 - T') & -I' & 0 \\ L_Y & L_r & \frac{M}{P^2} \\ \frac{F''}{F'} & 0 & \frac{W}{P^2} \end{bmatrix} \begin{bmatrix} dY \\ dr \\ dP \end{bmatrix} = \begin{bmatrix} dG \\ \frac{dM}{P} \\ \frac{dW}{P} \end{bmatrix}$$

which gives us something to which Cramer's rule is readily applicable. Remember the rule states that if you have a system $A\mathbf{y} = \mathbf{d}$ the solution value for y_j, the j^{th} element of the vector $\mathbf{y}$ of unknowns, is given by

$$y_j = \frac{|A_j|}{|A|}$$

where $|A_j|$ is the determinant of the matrix created by substituting the vector $\mathbf{d}$ into the j^{th} column of A.

In this case we have (finding the determinant by multiplying terms along all the diagonals, and giving the products the right signs)

$$\begin{aligned} |A| &= +[1 - C'(1 - T')]\,L_r \left(\frac{W}{P^2}\right) - 0 \\ &\quad +(-I')\left(\frac{M}{P^2}\right)\left(\frac{F''}{F'}\right) - (-I')\,L_Y\left(\frac{W}{P^2}\right) \\ &\quad +0 - 0 \\ &= [1 - C'(1 - T')]\frac{W}{P^2}L_r + I'\left(\frac{W}{P^2}L_Y - \frac{M}{P^2}\frac{F''}{F'}\right) \end{aligned}$$

Now suppose we want to solve the system for dY. In that case we substitute the $\mathbf{d}$ vector into the first column of A, yielding

$$A_j = \begin{bmatrix} dG & -I' & 0 \\ \frac{dM}{P} & L_r & \frac{M}{P^2} \\ \frac{dW}{P} & 0 & \frac{W}{P^2} \end{bmatrix}$$

in which case

$$\begin{aligned} |A_j| &= +dG\,L_r\left(\frac{W}{P^2}\right) - 0 \\ &\quad +(-I')\left(\frac{M}{P^2}\right)\left(\frac{dW}{P}\right) - (-I')\left(\frac{dM}{P}\right)\left(\frac{W}{P^2}\right) \\ &\quad +0 - 0 \\ &= \frac{W}{P^2}L_r\,dG + I'\frac{W}{P^2}\frac{dM}{P} - I'\frac{M}{P^2}\frac{dW}{P} \end{aligned}$$

The solution is therefore

$$dY = \frac{\frac{W}{P^2}L_r\,dG + I'\frac{W}{P^2}\frac{dM}{P} - I'\frac{M}{P^2}\frac{dW}{P}}{[1 - C'(1-T')]\frac{W}{P^2}L_r + I'\left(\frac{W}{P^2}L_Y - \frac{M}{P^2}\frac{F''}{F'}\right)}$$

To obtain the reduced-form multiplier with respect to dG we invoke *ceteris paribus* and set the other exogenous changes, dM and dW, to zero. Thus we're asking, what will be the effect of a small change in government spending on the equilibrium value of output, assuming no change in the money supply or the nominal wage?

The multiplier in question is

$$\frac{dY}{dG} = \frac{1}{[1 - C'(1-T')] + \frac{I'}{L_r}\left(L_Y - \frac{M}{P}\frac{F''}{F'^2}\right)}$$

Is it possible to make economic sense of this coefficient? Yes, I think so. Notice first that if investment spending is totally unresponsive to the interest rate (i.e., $I' = 0$), then this coefficient reduces to the simple multiplier with which you should be familiar from intermediate macro, namely

$$\frac{1}{1 - C'(1-T')}$$

Now, what difference does it make if $I' < 0$? First, L_r is presumed to be negative, so I'/L_r is positive. Within the parentheses at the right of the denominator of the multiplier, L_Y is positive and F'' is negative (while of course M, P and F'^2 are all positive). It follows that the whole term that depends on I' is positive. That is, the responsiveness of investment to the interest rate adds an extra positive component into the denominator of the reduced-form multiplier dY/dG, reducing the overall magnitude of that multiplier—or in other words, making fiscal policy less effective.

Why? Due to the 'crowding out' effect. The stimulus to GDP afforded by an increase in government spending results in an increase in the demand for money, which drives up the rate of interest, in turn reducing the level of investment. Note the parameters that 'modulate' this effect. The greater the value of L_Y, i.e., the greater the increase in money demand stemming from an increase in aggregate income, the more the crowding out. Crowding out is also exacerbated if F'' is strongly negative (that is, if the marginal return to labor is sharply diminishing), since in that case the increase in GDP stemming from the increase in G must be accompanied by a relatively substantial rise in P (to depress W/P in line with labor's lower marginal product). On the other hand, the greater the absolute value of L_r, the less the crowding out. If the interest-responsiveness of the demand for money is great, it takes only a relatively small increase in r to maintain money-market equilibrium in face of an increase in demand stemming from a rise in incomes.

Exercises

1. Isolate the reduced form coefficient dY/dW. Ascertain the sign of the effect and give an interpretation as offered above for dY/dG.

2. Apply Cramer's rule to find the solution for dP. Write out the reduced-form coefficients dP/dG, dP/dM and dP/dW, and give an account of the economic relationships they represent.

3. In place of taking money supply, M, as an exogenous variable, let the monetary authority behave according to the rule $M = M(r)$ with $M' > 0$. That is, the money supply is adjusted in reponse to changes in the rate of interest, in such a way as to stabilize the latter. Add this function to the initial seven equations and recalculate the model. Derive the reduced-form coefficient dY/dG and comment on the difference between this and the original version of dY/dG discussed above.