Economics 211 Allin Cottrell

Introduction to Linear Dynamic Models

1 A globally stable system

Take a system in which the time derivatives of the two variables x and y are functions of the levels of both x and y.

$$\frac{dx}{dt} \equiv \dot{x} = ax + by + h \tag{1}$$

$$\frac{dy}{dt} \equiv \dot{y} = cx + dy + k \tag{2}$$

Let the signs of the coefficients be as follows: $a < 0$, $b > 0$, $c < 0$, $d < 0$, $k > 0$ (the sign of h is immaterial).

Consider the locus of points in x, y space such that $\dot{x} = 0$, the *stationary locus* for x. It is obtained by setting the left-hand side of (1) to zero, which yields

$$y = -\frac{a}{b}x - \frac{h}{b}$$

Given the signs of the coefficients, the slope of this locus, $-a/b$, is positive: the higher is the value of x, the greater must be the value of y to hold x constant (i.e. to hold $\dot{x} = 0$).

Consider the thought experiment of taking a small step off the stationary locus for x in the x direction (as from A to B in Figure 1). At point A, $dx/dt = 0$ (by construction, since we're on the $\dot{x} = 0$ locus). At point B, y remains unchanged but the value of x is increased relative to A. Look at equation (1) and remember that $a < 0$: whenever x is increased but y is held constant, $\dot{x}$ must fall. But this means that $\dot{x}$ must be negative at B, which in turn means that x is falling over time, and we tend to move back toward the stationary locus.

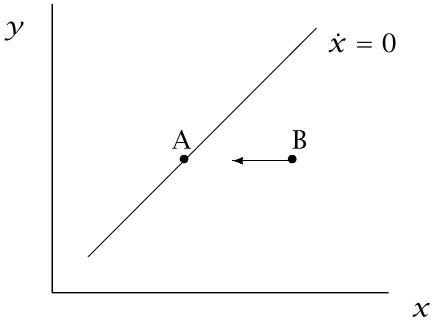

Figure 1:
Stationary locus for x

The stationary locus for y is derived in the same manner. It is represented by

$$y = -\frac{c}{d}x - \frac{k}{d}$$

and has negative slope $-c/d$. Repeating the above thought experiment we have the situation shown in Figure 2. Looking at equation (2) and remembering that $d < 0$ we can infer that a vertical jump from A to B will make $\dot{y}$ go negative, so that y falls and we tend to return to the stationary locus.

Putting two and two together we get the setup shown in Figure 3. This phase diagram—as it is called—exhibits *global stability*. The intersection of the two stationary loci represents the equilibrium point for the system, where both x and y remain unchanging. *The arrows of motion are such that wherever we start out in the phase space, we are always drawn inexorably toward the equilibrium.*

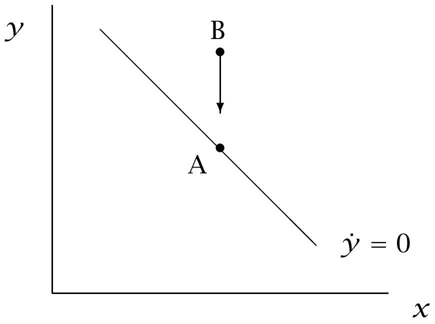

Figure 2:
Stationary locus for y

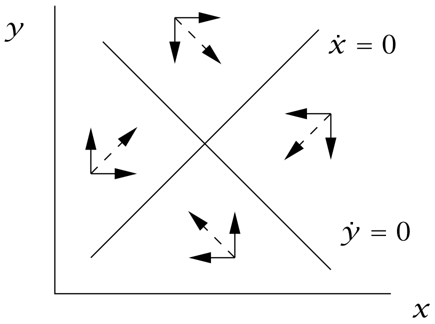

Figure 3:
This system is globally stable

2 An interpretation: IS–LM

Consider this simple linear IS–LM model:

$$\begin{aligned} L &= (l_1 y + l_2 r)P \qquad l_1 > 0,\, l_2 < 0 && (3)\\ I &= i_0 + i_1 r \qquad i_0 > 0,\, i_1 < 0 && (4)\\ S &= sy \qquad 0 < s < 1 && (5) \end{aligned}$$

The endogenous variables are y (real output), I (real investment), S (real saving), L (the demand for money) and r (the rate of interest). The exogenous variables are M (nominal money stock) and P (the general price level).

In equilibrium, the model is closed by specifying that $I = S$ and $L = M$, but suppose that when the system is out of equilibrium the variables r and y obey the following dynamic equations:

$$\begin{aligned} \dot{r} &= k(L - M) \qquad k > 0 && (6)\\ \dot{y} &= h(I - S) \qquad h > 0 && (7) \end{aligned}$$

This is to say that the interest rate is driven up (down) by an excess demand (supply) of money, while real output is driven up (down) by an excess (shortfall) of investment in relation to saving. This seems like a plausible set of dynamics.

Now let us analyse the disequilibrium dynamics of the system. Using (3) in (6) yields

$$\dot{r} = Pkl_2 r + Pkl_1 y - kM \tag{8}$$

while using (4) and (5) in (7) gives

$$\dot{y} = hi_1 r - hsy + hi_0 \tag{9}$$

Note that equations (8) and (9) are in the same form as the generic equations of motion (1) and (2) above: the time derivatives of r and y depend in a linear manner on the levels of r and y. The coefficients of the generic equations of motion correspond to those of our disequilibrium dynamic equations for r and y as shown in the following table. By inspecting the sign specifications in equa-

Generic		*IS–LM*
a	$\longleftrightarrow$	Pkl_2
b	$\longleftrightarrow$	Pkl_1
c	$\longleftrightarrow$	hi_1
d	$\longleftrightarrow$	$-hs$

tions (3)–(7) we can see that for the IS–LM dynamics we have: $Pkl_2 < 0$ (equivalent to $a < 0$), $Pkl_1 > 0$ ($b > 0$), $hi_1 < 0$ ($c < 0$) and $-hs < 0$ ($d < 0$). This sign pattern corresponds to that assigned in relation to equations (1) and (2). It follows that the IS–LM disequilibrium system provides an interpretation (or 'model', in the mathematical sense of the term) for the globally stable system of section 1. Wherever the system happens to 'start from' in r, y space, it will tend to evolve towards an equilibrium in which r and y remain unchanging.

3 Globally unstable systems

Return to equations (1) and (2) and consider the consequences of the following sign pattern for the coefficients: $a > 0$, $b < 0$, $c > 0$, $d > 0$. If you repeat the reasoning accompanying Figures 1 and 2 (do it!) you will see that if we 'step off' the stationary locus for x in the x direction, then according to equation (1) we tend to move further away from this locus, and similarly for the y stationary locus. The resulting situation is shown in Figure 4. In this case, for any starting point other than the equilibrium itself, the system will tend to diverge progressively from the equilibrium over time. Note that for an economic model with this property comparative statics would have little meaning. We could determine the change in the equilibrium values of the endogenous variables that results from a change in one or other of the parameters, but this would be somewhat pointless as the system will never reach its new equilibrium following such a disturbance.

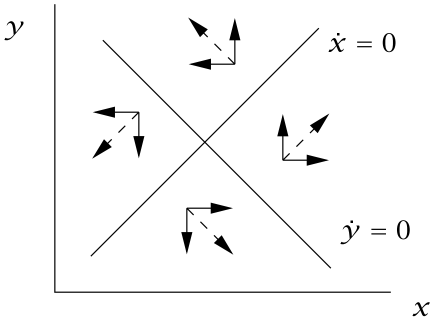

Figure 4:
This system is globally unstable

4 The saddlepoint property

An interesting situation arises when the sign pattern for the coefficients in (1) and (2) is as follows: $a > 0$, $b < 0$, $c < 0$, $d < 0$. The signs of a and b are the same as those specified in section 3 above, which means that the arrows of motion point away from $\dot{x} = 0$ locus in the x dimension. The signs of c and d, however, are the same as those in section 1, so the arrows of motion point toward the $\dot{y} = 0$ locus in the y dimension as shown in Figure 2. The resulting system is displayed in Figure 5. From any starting point to the left of $\dot{x} = 0$ and below $\dot{y} = 0$ the resultant motion is unequivocally away from the equilibrium; the same goes for starting points that are to the right of $\dot{x} = 0$ and above $\dot{y} = 0$. In the subspace to the left of $\dot{x} = 0$ and above $\dot{y} = 0$, the resultant motion—downward and to the left—*can* take us toward the equilibrium, if the starting point is chosen carefully. Similarly for the subspace to the right of $\dot{x} = 0$ and below $\dot{y} = 0$: motion upward and to the right will take us in the direction of equilibrium, for certain specific starting points. The set of points in the latter two subspaces that

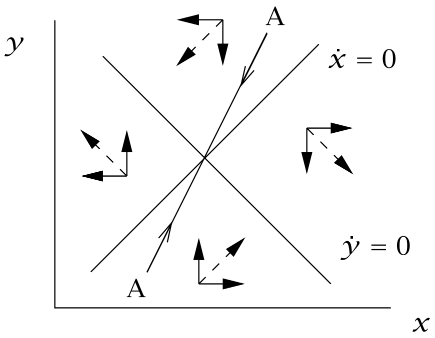

Figure 5:
This system has the saddlepoint property

satisfy this condition—i.e. if we start from any of them we will head toward equilibrium—is called the *convergent locus*; it is shown as the line AA in Figure 5.

The connection with a horse's saddle is clear if you think about it. In one dimension (namely, from the horse's head to its tail) a saddle slopes down towards the middle. This corresponds to the *stable stationary locus* ($\dot{y} = 0$ in the mathematical example). But in another dimension (flank to flank) the saddle slopes upward toward the middle. This corresponds to the unstable stationary locus, $\dot{x} = 0$.

While globally unstable models are basically uninteresting from the point of view of equilibrium analysis, we shall see that the saddlepoint system has some quite interesting macroeconomic interpretations. Such systems are of particular interest under the assumption of perfect foresight (or its stochastic counterpart, 'rational expectations'). Suppose we have a two-variable macro system where one of the variables is 'sticky' for some reason or other while the other is free to 'jump' to a new value at a point in time. Suppose an existing equilibrium state is disturbed, that is, the system's equilibrium point is moved to a new location in the phase space. Under what condition will the system proceed toward the new equilibrium? This will happen only if the system is somehow placed on the unique convergent path (corresponding to AA in Figure 5). If one variable is sticky and the other free to jump, the 'jump variable' will have to do the job of placing the system on the convergent path.

We will take a look at two models where this sort of effect is important.